CONVECTIVE INSTABILITY OF THERMAL WAVES NEAR BOUNDARIES

S. V. Dobkin and É. E. Son

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1. Introduction. In a medium with a nonlinear thermal conductivity, thermal waves with a sharp front can exist [1]. If the thermal diffusivity of the medium $\chi \sim T^n$ (T is the temperature), then the temperature gradient near the front of the thermal wave will be of the form $dT/dx \sim |x_f - x|^{1/n-1}$. The temperature in the thermal wave front decreases and the density of the gas medium correspondingly increases. If the thermal wave propagates upward in the field of gravity (directed downward), then a convective instability can arise. The effect of gravity on the hydrodynamic instability of a flare front was studied in [2]. Analogous effects occur when the acceleration is due to other causes, for example incidence of a shock or acoustic wave on the flare front; this is one of the causes of vibrational combustion [3]. The acceleration also affects the stability of the flare when it propagates with a variable velocity [2]. The effect of acceleration of the front on the Rayleigh–Taylor instability of the boundary between the detonation products and the gas in the case of a spherical explosion was discussed in [4]. The instability mechanism was related to the deceleration of the shock wave front and the passing of the detonation products out through the front, where the latter has a large density in comparison to that of the ambient gas.

A convective instability of thermal waves occurs when the acceleration of the front is positive, since, in this case, the inertial force in a coordinate system moving with the front is directed toward the gas with the smaller density. In an infinite medium such an instability cannot occur. For example, for the case of propagation of thermal waves from an instantaneous plane source, the acceleration of the front is negative for the dependence $\chi \sim T^n$ [1]:

$$g = \frac{d^2 x_f}{dt^2} \sim -\frac{n+1}{n+2} t^{-\frac{n+3}{n+2}}.$$

For a half-space with a constant temperature T_0 on the boundary we have $g \sim T^{n/2}_0 4t^{3/2}$ [1], and for a constant heat flux on the boundary $g \sim -2(n+1)/(n+2)^2 t^{-\frac{n+3}{n+2}}$.

A convective instability of thermal waves in a half-space is possible initially, until the self-modeling regime occurs. For example, consider the case where a cold gas is initially thermally insulated from a hot wall. After removal of the thermal insulation, the gas begins to heat up and for a nonlinear thermal diffusivity a thermal wave arises, which is decelerated at large values of the time, in correspondence with the self-modeling regime. But initially the gas is accelerated close to the wall, and this can cause a convective instability.

An analogous situation occurs for cooling waves. If the thermal diffusivity of the mediu decreases with increasing temperature ($\chi \sim T^{-n}$), then

$$g \sim \frac{n-1}{(n-2)^2} t^{-\left(\frac{n-3}{n-2}\right)}, \tag{1.1}$$

and hence at asymptotically large values of the time, the cooling wave is accelerated (in contrast to the deceleration of the heating wave) and the magnitude of the acceleration decreases. The solution (1.1) is invalid for the early stages of the self-modeling solution. For cooling near a wall, the gas flows into a cold region and slows down; therefore, a convective instability can arise.

Cooling waves are formed in cases when the thermal diffusivity of the medium decreases with increasing temperature. For a high-temperature gas this can occur through the mechanism of radiative thermal conductivity [1], which occurs as a result of a sharp decrease of the Rosseland passage of the radiation with increasing temperature. Another possibility is the

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formation of cooling waves in a temperature region in which there are chemical reactions (dissociation or ionization of the gas) and where the contribution of these chemical reactions to the thermal conductivity is decreasing.

Although the convective instability considered here occurs in a thin layer near the boundary, it can lead to the formation of a turbulent region and to more intense heat exchange between the gas and wall. For a high-temperature gas, where the existence of cooling waves depends on the radiative thermal conductivity, this mechanism can be significant for the problems of reflection and interaction of shock waves with surfaces (the ends of shock tubes).

We consider the instability of a gas occupying a half-space and cooled by an unbounded region. This model describes experiments of the following type: At the initial instant of time let a semiinfinite pipe be filled with hot gas of temperature T_0, which either is at rest or flows into a cooled wall with a constant velocity, corresponding to reflection of a shock wave from the wall. Upon contact with the wall, the gas begins to cool, and the gas moves toward the cooled wall of the pipe, which causes the gas to cool, since close to the wall the gas is decelerated. Therefore, the acceleration of the gas changes sign in the region of motion. The resolution time of the pressure nonuniformities of an acoustic wave is less than the characteristic transit time of the thermal wave; therefore, the pressure can be assumed to be constant, which means that the density profile will be inversely related to the temperature profile, i.e., near the wall the gas is denser.

For certain dependences of the thermal diffusivity on temperature a condition is possible in which the acceleration of the cooled gas is in the direction of higher gas densities, and this leads to convective instability of the gas. For accelerations exceeding a certain critical value, turbulent motion of the gas near the face of the wall can arise, which smooths out the temperature distribution of the gas near the wall.

The problem consists of three parts: It is necessary to find the unperturbed motion of the gas, which is assumed to be laminar and nonstationary; it is necessary to study the stability of this motion and to consider the dynamics of the development of the turbulent region.

2. Unperturbed Motion of the Gas in a Cooling Wave. The system of equations describing the one-dimensional nonstationary motion of the gas includes the equation of continuity, the equations of momentum and energy balance, and the equation of state of the gas:

$$\frac{d\rho}{dt} + \rho\frac{\partial v}{\partial x} = 0; \tag{2.1}$$

$$\rho\frac{dv}{dt} = -\frac{\partial p}{\partial x} + \frac{4}{3}\frac{\partial}{\partial x}\left(\eta\frac{\partial v}{\partial x}\right); \tag{2.2}$$

$$\rho c_p\frac{dT}{dt} = \frac{dp}{dt} + \frac{4}{3}\eta\left(\frac{\partial v}{\partial x}\right)^2 + \frac{\partial}{\partial x}\left(\lambda\frac{\partial T}{\partial x}\right); \tag{2.3}$$

$$p = \rho RT. \tag{2.4}$$

Here η is the viscosity, $\lambda(T)$ is the thermal conductivity, and c_p is the heat capacity of the gas, and $x > 0$.

Neglecting viscous dissipation in comparison with heat transport ($\eta(\partial v/\partial x)^2 \ll \lambda|\partial^2 T/\partial x^2|$) and assuming the pressure is constant (since $La_s/\kappa \gg 1$, where L is the characteristic linear dimension of the nonuniformities and a_s is the speed of sound), we transform to Lagrangian coordinates.

The equations of continuity and energy (for c_p = const) in Lagrangian coordinates (a) have the form

$$\rho dx = \rho_0 da; \tag{2.5}$$

$$\frac{\partial T(a,t)}{\partial t} = \frac{\partial}{\partial a}\left(\frac{\lambda}{\rho_0 c_p}\frac{\rho}{\rho_0}\frac{\partial T}{\partial a}\right). \tag{2.6}$$

For a constant pressure, the density and thermal conductivity of the gas are determined by the temperature; therefore, the energy equation (2.6) can be solved for given initial and boundary conditions.

Introducing the variable $\xi = a/2(\chi_0 t)^{1/2}$, we write (2.6) in the form

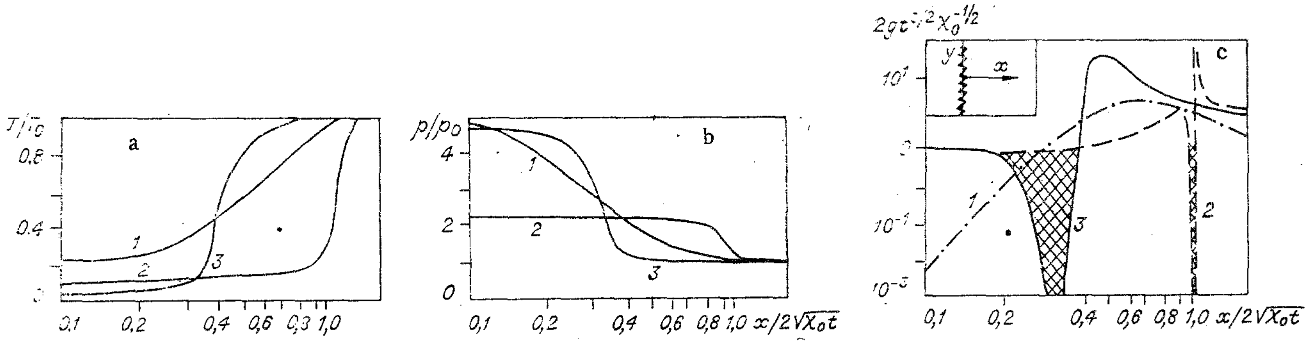

Fig. 1

$$\frac{d}{d\xi} f(\theta) \frac{d\theta}{d\xi} + 2\xi \frac{d\theta}{d\xi} = 0, \tag{2.7}$$

where $\theta = T/T_0$, $f(\theta) = \rho(\theta)\lambda(\theta)/\rho_0\lambda_0$, $\chi_0 = \lambda_0/\rho_0 c_p$.

We assume that the gas is bounded by an infinite wall whose temperature at $x = -\infty$ is held constant $[T(-\infty) = T_w]$. The equation of heat propagation inside the wall is

$$\frac{\partial T}{\partial t} = \chi_w \frac{\partial^2 T}{\partial x^2}, \quad x < 0, \tag{2.8}$$

where χ_w is the thermal diffusivity of the wall.

The boundary conditions for (2.3) and (2.8) follow from the continuity of temperature and heat flux at the wall:

$$\lambda_w \frac{\partial\theta}{\partial x}\bigg|_{x=-0} = \lambda \frac{\partial\theta}{\partial x}\bigg|_{x=+0}, \quad \theta(-0) = \theta(+0). \tag{2.9}$$

Transforming to Lagrangian coordinates, we obtain the boundary conditions for (2.7):

$$\frac{d\theta}{d\xi}\bigg|_{\xi=0} = \frac{m(\theta - \theta_w)}{f(\theta)}\bigg|_{\xi=0}, \quad m = \left(\frac{4\lambda_w \rho_w c_w}{\pi\lambda_0\rho_0 c_p}\right)^{1/2}, \quad \theta(\infty) = 1. \tag{2.10}$$

The connection between the Euler and Lagrangian coordinates is given by the equation of continuity (2.5):

$$\frac{x}{2(\chi_0 t)^{1/2}} = \int_0^{\xi(\theta)} \frac{\rho_0}{\rho(\xi')} d\xi'. \tag{2.11}$$

The velocity v and acceleration g of the gas can be found by differentiating (2.11), since the viscosity is zero for one-dimensional motion:

$$v(\xi, t) = \frac{1}{2}\left(\frac{\chi_0}{t}\right)^{1/2}\left[f(\theta)\frac{d\theta}{d\xi} - f(0)\frac{\partial\theta}{\partial\xi}\bigg|_{\xi=0}\right]; \tag{2.12}$$

$$g(\xi, t) = -\frac{v}{2t} + \frac{\xi^2 \chi_0^{1/2}}{2t^{3/2}} \frac{d\theta}{d\xi}. \tag{2.13}$$

Equation (2.7) can be solved analytically only for the case $\lambda \sim T^n$, n = 0, 1, 2 [5]. For an arbitrary form of $f(\theta)$ it must be solved numerically. Calculations of the temperature and acceleration profiles of the cooled gas show that a region of convective instability is formed when the thermal conductivity of the gas decreases rapidly enough with increasing temperature; in this case, the cooling waves have a sharp front. The existence of a front to the cooling waves for this type of $\lambda(T)$ dependence is supported by analysis similar to that leading to the existence condition for heating waves [1].

The numerical calculations show that for a temperature dependence of the thermal conductivity of the form $\lambda \sim T^{-n}$, a front with a region of convective instability forms for $n \geqslant 2.5$.

In Fig. 1a-c we show the dimensionless profiles of temperature, density, velocity, and acceleration of the gas as functions of the exponent n = 1, 2.5, 4.5 (curves 1-3, respectively). When $n \geqslant 2.5$, the cooling wave front becomes sharper and the region of gas in which convective instability is possible increases in size.

3. Stability of the Unperturbed Motion. In order to study the convective instability of the one-dimensional unperturbed motion of the gas, it is necessary to consider three-dimensional perturbations. A nonuniform perturbation developing along the y axis (insert in Fig. 1c) can have an arbitrary spectrum; therefore, the problem is complicated in the case when the characteristic linear dimension of the nonuniformity along the y axis is of the same order of magnitude as the linear dimension of the nonuniformity of the unperturbed nonstationary motion. It is important, however, to show that this type of laminar motion of the cooled gas is unstable. Hence, if we are not interested in obtaining the critical condition for transition to turbulent motion, we may consider only the short-wavelength perturbations. In this case the quasistationary approximation is used, in which we will assume that the characteristic scale of variation of the perturbation is smaller than the scale of the unperturbed, nonstationary problem (the perturbation is considered to be a "ripple" on a slowly varying (in space) distribution of unperturbed parameters). Since the perturbed motion is three-dimensional, the advantage of a Lagrangian description is lost, and we therefore use the Euler description of the motion.

Assuming subsonic perturbations, the equation of continuity can be taken in the Boussinesq approximation div $\mathbf{v}_1 = 0$ [6]. Here and below we let the subscript 1 refer to perturbations of the various parameters. Then the linearized equation of motion has the form

$$\rho_1 \frac{\partial \mathbf{v}}{\partial t} + \rho \frac{\partial \mathbf{v}_1}{\partial t} = -\nabla p_1 + \rho\nu\Delta \mathbf{v}_1. \tag{3.1}$$

We apply the operation rot rot to (3.1) and take the x component of the result. Assuming the kinematic viscosity ν is constant and the unperturbed equation is quasistationary, we obtain

$$\frac{\partial}{\partial t}\Delta v_1 = \frac{g}{T}\Delta_\perp T_1 + \nu\Delta\Delta v_1, \quad \Delta_\perp \equiv \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \tag{3.2}$$

The linearized equation of heat conduction in the quasistationary approximation is

$$\frac{\partial T_1}{\partial t} + v_1 \frac{\partial T}{\partial x} = \chi \Delta T_1. \tag{3.3}$$

The system of equations (3.2) and (3.3) reduces to the equations of thermal convection if we replace the acceleration of gravity by the acceleration of the gas at a given point. Solving the resulting system of equations by the method of Galerkin with Rayleigh boundary conditions [6] (since the convective instability criterion does not depend very significantly on the exact boundary conditions [6]) we find the following expression for the perturbation increment γ:

$$\gamma(k) = -\frac{\alpha(\nu+\chi)}{2} \pm \sqrt{\frac{\alpha^2}{4}(\nu-\chi)^2 - \frac{gk^2}{\alpha T}\frac{\partial T}{\partial x}}. \tag{3.4}$$

Here $\alpha = m^2\pi^2/L^2$, m is an integer, k is the wave number, L is the characteristic dimension of the nonuniformities, which is of the order of the linear dimension of the region in which $g < 0$. For short-wave perturbations ($kL \gg 1$) and sufficiently large values of the parameter $g/T(\partial T/\partial x)$, convective instability arises.

4. Effect of Turbulence on the Cooling of the Gas. Turbulent mixing in a region where there is an instability of the Taylor type was studied in [7-9], where it was shown that the scale of turbulence is $\ell = \alpha L$ (L is the linear dimension of the region of mixing, and α = 0.1-0.4). In the problem considered here, the turbulent region rapidly expands to the dimension of the region where the acceleration of the gas is negative. A perturbation propagating from this region into a region with positive acceleration is damped out because this direction of acceleration stabilizes the perturbation. Because of this effect, the temperature distribution in the region with negative acceleration will be smoothed out. Intense heat exchange in a region with a sharp temperature gradient can lead to intense cooling of the gas through the instability region with small thermal resistance.

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NONLINEAR WAVES AND STABILIZATION OF TWO-DIMENSIONAL INSTABILITY IN A BOUNDARY LAYER

V. P. Reutov

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As a rule, the transition to turbulence in a boundary layer is associated with the growth of two-dimensional waves [1-4]. Consequently, the investigation of the nonlinear stage in the development of a two-dimensional instability has an important role in the creation of a transition theory. Methods of the theory of a weak nonlinearity permit computation of the coefficients in the dynamic equations proposed by Landau [5, 6] for the weak wave amplitude. However, for those values of the flow parameters that are ordinarily realized in the transition region, the weak nonlinearity approximation describes only the initial stage of wave amplification. Substantially nonlinear structures that originate in the boundary layer because of the constraint of the two-dimensional instability are examined in this paper.

The mechanism of boundary-layer instability in the case of infinitesimally small perturbations has long been studied (see [1, 7], say). It is known that the occurrence of a viscous near-wall layer (VNWL) results in wave destabilization, while resonance wave-flow interaction can attenuate or totally suppress this instability. In the case when the thickness of the resonance domain of flow interaction with the wave, the critical layer (CL) is sufficiently small, there is a possibility of analytical investigation of the substantially nonlinear stage in development of the instability [8-12]. Simplification of the problem is associated with localization of the nonlinearity within the limits of a thin CL. However, formation of the VNWL was not taken into account in [8] (slip conditions were posed at the wall). The shift in the primary flow velocity near the wall was not taken into account in [9] in the determination of the VNWL structure, which is only justified for very large Reynolds numbers. Moreover, it follows from the solution of the nonstationary problem [12] that the natural waves constructed in [9] correspond to the threshold of strict origination of instability (and are not constrained by it). Below, we solve the problem of stationary waves originating for moderately large Reynolds numbers that are characteristic for the main part of the boundary layer neutral curve loop. The analysis is constructed within the framework of CL theory and is based on graphic representations of the CL structure and the instability mechanism. From the formal point of view, the procedure proposed for the solution can be considered a generalization of the Tollmien method used to construct the neutral curve in the linear theory of hydrodynamic instability [1]. The results of computations are compared with known experimental data.

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